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On Rational Exuberance*

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Abstract

In his seminal contribution, Tirole (1985) shows that an overlapping generations economy may monotonically converges to a steady state with a positive rational bubble, characterized by the dynamically efficient golden rule. The issue we address is whether this monotonic convergence to an efficient long-run equilibrium may fail, while the economy experiences persistent endogenous fluctuations around the golden rule. Our explanation leads on the features of the credit market. We consider a simple overlapping generations model with three assets: money, capital and a pure bubble (bonds). Collateral matters because increasing his portfolio in capital and bubble, the household reduces the share of his consumption paid by cash. From a positive point of view, we show that the bubbly steady state can be locally indeterminate under arbitrarily small credit market imperfections and, thereby, persistent expectation-driven fluctuations of equilibria with (rational) bubbles can arise. From a normative point of view, monetary policies that are not too expansive, are recommended in order to rule out the occurrence of sunspot fluctuations and enhance the welfare evaluated at the steady state.

Keywords: Bubbles, collaterals, indeterminacy, cash-in-advance constraint, overlapping generations.

Résumé

Dans son influente contribution, Tirole (1985) montre qu'une économie à générations imbriquées peut converger de façon monotone vers un état stationnaire avec une bulle rationnelle, caractérisé par la règle d'or dynamiquement efficace. La question qui nous intéresse est d'étudier si cette convergence monotone vers un état de long terme efficace peut être remise en cause, l'économie connaissant des fluctuations endogènes persistentes autour de la règle d'or. Notre explication est basée sur les caractéristiques du marché du crédit. Nous considérons un modèle à générations imbriquées simple avec trois actifs : la monnaie, le capital et une bulle (obligation). Le collatéral joue un rôle parce qu'en augmentant son portefeuille en capital et bulle, le consommateur réduit la part de sa consommation financée par la monnaie. D'un point de vue positif, nous montrons que l'état stationnaire avec bulle peut-être localement indéterminé pour de faibles imperfections du marché du crédit et, ainsi, des fluctuations dues à la volatilité des anticipations avec une bulle (rationnelle) peuvent émerger. D'un point de vue normatif, des politiques monétaires qui ne sont pas trop expansives sont recommendées pour éliminer les fluctuations dues aux anticipations auto-réalisatrices et augmenter le bien-être à l'état stationnaire.

Mots-clés: Bulles, collatéral, indétermination, contrainte de liquidité, générations imbriquées.

JEL classification: D91, E32, E50.

1 Introduction

" [...] Clearly, sustained low inflation implies less uncertainty about the future, and lower risk premiums imply higher prices of stocks and other earning assets. We can see that in the inverse relationship exhibited by price/earnings ratios and the rate of inflation in the past. But how do we know when *irrational exuberance* has unduly escalated asset values, which then become subject to unexpected and prolonged contractions as they have in Japan over the past decade? [...] " A. Greenspan, 1996.

These controversial words by Alan Greenspan, namely "irrational exuberance", have driven us to deepen the meaning of exuberance and focus on the existence and persistence of rational instead of irrational exuberance. In the following, we tackle this issue in a precise sense, by characterizing the existence of persistent expectation-driven fluctuations of a rational bubble.

The overlapping generations model provides an appropriate dynamic general equilibrium framework to prove the existence of rational bubbles. As shown by Tirole (1982, 1985), bubbles arise because agents are short-lived and have finite horizon.[1] In his seminal paper, Tirole (1985) explains that the existence of a bubbly steady state requires the coexistence of a dynamically inefficient bubbleless steady state. In addition, he proves that a unique equilibrium path converges monotonically to the efficient bubbly steady state.

Rational exuberance can be interpreted as exuberance of rational bubbles and exuberance, in turn, as volatility of expectations. This explains why we are interested in showing that rational bubbles can experience persistent expectation-driven fluctuations in a dynamic general equilibrium model. Moreover, we emphasize that the occurrence of such endogenous fluctuations rests on the features of credit market.

We extend the overlapping generations model proposed by Tirole (1985), where consumers save through assets representing productive capital and a pure bubble (bonds), by introducing money as a third asset. Money is needed for transactions as a mean of exchange:[2] a share of second-period consumption in a two-period life is paid by cash;[3] the rest is financed on non-monetary savings or credit (capital and bonds). An additional novel feature of the model is the assumption that the credit share of consumption purchases grows with the amount of non-monetary savings. This comes from a simple observation: because of public regulation or banking practices based on credit market imperfections such as asymmetric informations, a consumer owning more collaterals

[1] See also Tirole (1990) for an introductory survey.

[2] In our model, in contrast to several contributions (among the others, Michel and Wigniolle (2003, 2005) and Weil (1987)), the bubble is not purely monetary. On the one hand, real balances are valued because of their liquidity services and we focus on equilibria where the cash-in-advance constraint is binding. On the other hand, the bubble is a positive-priced bond with no fundamental value.

[3] See, in particular, Hahn and Solow (1995). The interested reader can refer to Crettez et al. (1999) who present various cash-in-advance constraints in overlapping generations model with capital accumulation à la Diamond.

(capital and bonds) can increase her/his credit opportunities and the corresponding share of consumption. Notice also that this goes in opposite direction of market distortions: the larger the collaterals, the lower the rationing degree on credit market.

Before characterizing the indeterminacy of the bubbly steady state, which corresponds to the golden rule, we study its properties, in particular the role of monetary policy. We show that a decrease in the money growth rate is welfare improving at the steady state.

Studying the local dynamics, we show that, under a constant credit share, the bubbly steady state is always determinate, but endogenous cycles of period two can emerge. Conversely, when collateral matters and the credit share increases with non-monetary savings, endogenous cycles may not only emerge, but the bubbly steady state can be indeterminate. In this case, persistent expectation-driven fluctuations of the rational bubble emerge, founding rational exuberance on a theoretical ground. It is also worthwhile to notice that these fluctuations appear for arbitrarily small distortions in the credit market[4] and are essentially explained by the opposite dynamic patterns for real balances and non-monetary savings. Finally, it is important to notice that a not too expansive monetary policy rules out indeterminacy, while as seen above it increases the stationary level of welfare.[5]

This issue of fluctuations of a rational bubble has been addressed in a few previous works. Weil (1987) shows the existence of sunspot equilibria, where the bubble can burst with positive probability. However, his analysis is based on a Markovian transition matrix, where probabilities are exogenous, and is inappropriate to explain persistent fluctuations of the bubble. In Azariadis and Reichlin (1996), endogenous fluctuations of the bubble (debt) may occur through a Hopf bifurcation. However, in contrast to our result, their analysis requires sufficiently large increasing returns,[6] i.e. strong market imperfections. Finally, Michel and Wigniolle (2003, 2005) provide an alternative history for bubbly fluctuations. Cycles between a bubbly regime (in terms of real balances) and a regime where the cash-in-advance constraint is binding are exhibited. Hence, fluctuations occurs, but in contrast to our findings, the bubble does not persist along the whole dynamic path.

The rest of the paper is organized as follows. In Section 2, we present the model, while, in Section 3, we define the intertemporal equilibrium. Section 4 is devoted to study the bubbly regime. In Section 5, we show the indeterminacy of a bubbly steady state. Section 6 concludes the paper, while many technical details are gathered in the Appendix.

[4] We mean a credit share close to one jointly with a small elasticity of the credit share with respect to non-monetary savings.

[5] Such a policy recommendation is in contrast to Michel and Wigniolle (2005) where a sufficiently expansive monetary creation avoids fluctuations between a regime with a bubble and a regime with a binding cash-in-advance constraint.

[6] Indeed, the real interest rate has to be increasing in capital.

2 The model

We consider an overlapping generations model with two-period lived households in discrete time, $t = 0, 1, ..., +\infty$.

2.1 Households

At period t, N_t individuals are born. Every one consumes an amount c_{1t} of final good and supplies inelastically one unit of labor when young, and consumes c_{2t+1} when old. Population growth is constant, $n \equiv N_{t+1}/N_t > 0$.

In order to ensure the consumption during the retirement age, people save through a diversified portfolio of nominal balances M_{t+1}, bonds B_{t+1} and productive capital K_{t+1}.[7] Bonds are remunerated at an interest rate, capital is used by firms to produce the consumption good, while money demand is rationalized by a cash-in-advance constraint in the second period of life. In the following, p_t will denote the price of consumption good, i_{t+1} and r_{t+1} the rental factors on bonds and capital, respectively, and w_t the real wage.

Preferences are summarized by a Cobb-Douglas utility function in consumption of both periods:

$$U(c_{1t}, c_{2t+1}) \equiv {c_{1t}}^a c_{2t+1}^{1-a} \tag{1}$$

with $a \in (0, 1)$.

The representative household of a generation born at time t derives consumption and assets demands (money, bonds and capital), by maximizing the utility function (1) under the first and second-period budget constraints:

$$\frac{M_{t+1}}{p_t N_t} + \frac{B_{t+1}}{p_t N_t} + \frac{K_{t+1}}{N_t} + c_{1t} \leq \tau_t + w_t \tag{2}$$

$$c_{2t+1} \leq \frac{M_{t+1}}{p_{t+1} N_t} + i_{t+1} \frac{B_{t+1}}{p_{t+1} N_t} + r_{t+1} \frac{K_{t+1}}{N_t} \tag{3}$$

where $\tau_t = (M_{t+1} - M_t) / (p_t N_t)$ are the monetary transfers distributed to young households. In addition, at the second period of life, each consumer faces a cash-in-advance constraint:

$$[1 - \gamma(s_t)] p_{t+1} c_{2t+1} \leq \frac{M_{t+1}}{N_t} \tag{4}$$

where s_t represents the non-monetary savings:

$$s_t \equiv \frac{B_{t+1}}{p_t N_t} + \frac{K_{t+1}}{N_t}$$

When the cash-in-advance constraint is binding, a share $1 - \gamma(s) \in (0, 1)$ of consumption purchases has to be paid cash.[8] The remaining part $\gamma(s)$ can be

[7] We assume a full capital depreciation within a period.

[8] We take in account a criticism addressed to the cash-in-advance literature: money velocity $1/[1 - \gamma(s)]$ is endogenous and no longer constant.

paid at the end of the period and denotes the credit share, that is the fraction of consumption good bought on credit. Individual non-monetary savings s_t works as collateral in order to reduce the need of cash, i.e. the larger the collaterals, the easier the purchasing on credit.[9]

The shape of credit share γ can be viewed as a restriction due to lenders' or sellers' prudential attitude towards borrowers in presence of asymmetric informations, but also as a credit market regulation policy, that is a legal constraint to credit grants in order to ensure borrowers' solvability.

Assumption 1 $\gamma(s) \in (0,1)$ *is a continuous function defined on* $[0,+\infty)$, C^2 *on* $(0,+\infty)$ *and strictly increasing* ($\gamma'(s) > 0$). *In addition, we define:*

$$\begin{aligned}\eta_1(s) &\equiv \frac{\gamma'(s)s}{\gamma(s)},\ \eta_2(s) \equiv \frac{\gamma''(s)s}{\gamma'(s)} \\ \eta_\eta(s) &\equiv \frac{\eta_1'(s)s}{\eta_1(s)} = 1 - \eta_1(s) + \eta_2(s)\end{aligned} \tag{5}$$

We note that when $\eta_1(s) = 0$ and γ tends to 1, money is no longer needed and the credit market distortion disappears. Our framework collapses in the seminal model by Tirole (1985).

Defining the inflation factor as $\pi_{t+1} \equiv p_{t+1}/p_t$, we get a no-arbitrage condition as portfolio choice:

$$i_{t+1} = \pi_{t+1} r_{t+1} \tag{6}$$

Introducing the variables per capita $m_t \equiv M_t/(p_t N_t)$, $b_t \equiv B_t/(p_t N_t)$ and $k_t \equiv K_t/N_t$, constraints (2)-(4) write:

$$n\pi_{t+1}m_{t+1} + s_t + c_{1t} \leq \tau_t + w_t \tag{7}$$

$$c_{2t+1} \leq nm_{t+1} + r_{t+1}s_t \tag{8}$$

$$[1-\gamma(s_t)]\,c_{2t+1} \leq nm_{t+1} \tag{9}$$

where now

$$s_t = n(k_{t+1} + \pi_{t+1}b_{t+1}) \tag{10}$$

Each household maximizes (1) under the budget and cash-in-advance constraints (7)-(9), determines an optimal portfolio (m_{t+1}, s_t) and an optimal consumption plan (c_{1t}, c_{2t+1}).[10]

Assumption 2 *Let* $\omega_{t+1} \equiv s_t/(s_t + n\pi_{t+1}m_{t+1})$. *For all* $t \geq 0$, *we assume* $i_t > 1$ *and*

$$\eta_1(s_t) < \frac{1-\gamma(s_t)}{\gamma(s_t)} \frac{\omega_{t+1}}{1-\omega_{t+1}} \tag{11}$$

[9] In fact, we extend the cash-in-advance constraint proposed by Hahn and Solow (1995) to the case where the share of consumption when old paid by cash depends on non-monetary savings.

[10] We observe that households are aware of the credit share function and consider its argument s as a choice variable.

In contrast to Michel and Wigniolle (2003, 2005), we consider only a binding cash-in-advance constraint.

Lemma 1 *Under Assumption 2, constraints (7)-(9) are binding.*

Proof. See the Appendix.

In order to ensure the different constraints to be binding, we assume that money is a dominated asset, that is $r_{t+1} > 1/\pi_{t+1}$ or, equivalently, $i_{t+1} > 1$. The opportunity cost of holding money, that is the nominal interest rate $i_{t+1} - 1$, is supposed to be strictly positive. Moreover, inequality (11) puts an upper bound to the credit-share elasticity $\eta_1(s)$. In fact, if collaterals matter too much, people no longer hold money and the cash-in-advance constraint fails to be binding.

Let $R^s_{t+1} \equiv r_{t+1} - \gamma'(s_t)\, c_{2t+1}$ and $R^m_{t+1} \equiv 1/\pi_{t+1} - \gamma'(s_t)\, c_{2t+1}$. Under Assumption 2, solving the optimal households' behavior, we get:

$$\frac{U_1(c_{1t}, c_{2t+1})}{U_2(c_{1t}, c_{2t+1})} = \frac{1}{\pi_{t+1}} \frac{R^s_{t+1}}{\gamma(s_t)\, R^m_{t+1} + [1-\gamma(s_t)]\, R^s_{t+1}} > \frac{1}{\pi_{t+1}} \tag{12}$$

where the last inequality holds because money is a dominated asset ($R^m_{t+1} < R^s_{t+1}$).[11] We further note that under a constant credit share ($\gamma(s) = \gamma$), equation (12) rewrites:

$$\frac{U_1(c_{1t}, c_{2t+1})}{U_2(c_{1t}, c_{2t+1})} = \frac{r_{t+1}}{1 + (1-\gamma)(i_{t+1} - 1)}$$

While the left-hand side is a marginal rate of intertemporal substitution, the right-hand side would reduce to r_{t+1} when γ tends to 1, as in the non-monetary model by Diamond (1965). In the limit case, there is no market distortion. When $\gamma < 1$, money demand entails an opportunity cost which lowers the real return on portfolio. More precisely, the household has to pay cash $1-\gamma$ to consume an extra-unit when old. The interest rate $i_{t+1} - 1$ on the cash holding entails an opportunity cost $(1-\gamma)(i_{t+1} - 1)$ which reduces the purchasing power of non-monetary saving. Further, when the credit share depends on collaterals, the marginal impact of savings on the credit share ($\gamma'(s) > 0$) becomes an additional distortion.

2.2 Firms

A competitive representative firm produces the final good using the constant returns to scale technology $f(K/N)\, N$, where the intensive production function $f(k)$ satisfies:

[11] Second order conditions are derived in the Appendix. We show that they are satisfied for $\eta_2(s) \leq 2(\eta_1(s) - 1)$ or $\eta_1(s)$ sufficiently low.

Assumption 3 *$f(k)$ is a continuous function defined on $[0, +\infty)$ and C^2 on $(0, +\infty)$, strictly increasing ($f'(k) > 0$) and strictly concave ($f''(k) < 0$). We further assume* $\lim_{k \to 0^+} f'(k) > n > \lim_{k \to +\infty} f'(k)$.

As usual, the competitive firm takes the prices as given and maximizes the profit $f(K_t/N_t) N_t - w_t N_t - r_t K_t$:

$$\begin{aligned} r_t &= f'(k_t) \equiv r(k_t) \\ w_t &= f(k_t) - k_t f'(k_t) \equiv w(k_t) \end{aligned} \tag{13}$$

For further reference, $\alpha(k) \equiv f'(k) k / f(k) \in (0, 1)$ will denote the capital share in total income and $\sigma(k) \equiv [f'(k) k / f(k) - 1] f'(k) / [k f''(k)] > 0$ the elasticity of capital-labor substitution. The interest rate and wage elasticities depend on $\alpha(k)$ and $\sigma(k)$:

$$\begin{aligned} \varepsilon_r(k) &\equiv \frac{r'(k) k}{r(k)} = -\frac{1 - \alpha(k)}{\sigma(k)} \\ \varepsilon_w(k) &\equiv \frac{w'(k) k}{w(k)} = \frac{\alpha(k)}{\sigma(k)} \end{aligned} \tag{14}$$

2.3 Monetary policy

A simple monetary policy is considered: money grows at a constant rate, $M_{t+1}/M_t = \mu > 0$. Focusing on real variables, we can decompose the money growth in the product of demographic growth, inflation and economic growth:

$$\mu = n \pi_{t+1} m_{t+1} / m_t \tag{15}$$

According to the Friedman's metaphor, money is helicoptered to young consumers by the monetary authority through lump-sum transfers $\tau_t = (M_{t+1} - M_t) / (p_t N_t)$ or, in real terms:

$$\tau_t = n \pi_{t+1} m_{t+1} - m_t \tag{16}$$

2.4 Bonds

Bonds follow $B_{t+1} = i_t B_t$.[12] Using real variables per capita, we get:

$$i_t b_t = n \pi_{t+1} b_{t+1} \tag{17}$$

Because they have zero intrinsic (fundamental) value, bonds are pure bubbles.

[12] For instance, one can assume that this asset is supplied by the government. b_t can be considered as a (real) engagement to repay b_t unit of consumption, whatever the price p_t. Alternatively, B_t can be interpreted as the (monetary) price of a quantity of asset normalized to one. In both the cases, B_t is a non-predetermined variable.

3 Equilibrium

Substituting (16) in the first-period budget constraint (7), we find:

$$m_t + s_t + c_{1t} = w(k_t) \tag{18}$$

where m_t represents the individual demand for real balances. [13] Using (8) and (9), we obtain:

$$\begin{aligned} m_{t+1} &= s_t \frac{r(k_{t+1})}{n} \frac{1-\gamma(s_t)}{\gamma(s_t)} & (19) \\ c_{2t+1} &= r(k_{t+1}) \frac{s_t}{\gamma(s_t)} & (20) \end{aligned}$$

Replacing (19) into (15), we deduce the inflation factor:

$$\pi_{t+1} = \frac{\mu}{n} \frac{\gamma(s_t)}{\gamma(s_{t-1})} \frac{1-\gamma(s_{t-1})}{1-\gamma(s_t)} \frac{r(k_t)\, s_{t-1}}{r(k_{t+1})\, s_t} \tag{21}$$

$>$From these expressions, we derive two equations that determine the dynamics of the economy. On the one side, from (12), (20) and (21), the consumers' intertemporal trade-off writes:

$$x_{t+1} = \frac{1-a}{a} \frac{[1-\eta_1(s_t)]\, s_t r(k_{t+1})}{\gamma(s_t)\, s_t + \mu[1-\gamma(s_t)-\eta_1(s_t)]\, s_{t-1} \frac{r(k_t)}{n} \frac{\gamma(s_t)}{1-\gamma(s_t)} \frac{1-\gamma(s_{t-1})}{\gamma(s_{t-1})}} \tag{22}$$

where

$$x_{t+1} \equiv \frac{c_{2t+1}}{c_{1t}} = \frac{s_t r(k_{t+1}) / \gamma(s_t)}{w(k_t) - s_t - s_{t-1} \frac{r(k_t)}{n} \frac{1-\gamma(s_{t-1})}{\gamma(s_{t-1})}} \tag{23}$$

is obtained from (18), (19) and (20).[14] On the other side, combining (6), (10) and (17) gives:

$$r(k_t)(s_{t-1} - nk_t) = n(s_t - nk_{t+1}) \tag{24}$$

Markets clear over time when these equations holds. More precisely:

Definition 1 *An intertemporal equilibrium with perfect foresight is a sequence* $(s_{t-1}, k_t) \in \mathbb{R}^2_{++}$, $t = 0, 1, ..., +\infty$, *such that (22)-(24) are satisfied, given* $k_0 = K_0/N_0 > 0$.

[13] Note that aggregating (8) and (18), and substituting (10) and (17), we recover the equilibrium in the goods market:

$$c_{1t} + c_{2t}/n + nk_{t+1} = r(k_t)\, k_t + w(k_t) = f(k_t)$$

[14] The positivity of the right-hand side of (22) is ensured by (11) (see the proof of Lemma 1). Hence, x_{t+1}, solution of (22), will be also positive at equilibrium.

Equations (22)-(24) constitute a two-dimensional dynamic system which determines from the initial condition the equilibrium path $(s_{t-1}, k_t)_{t\geq 0}$, where only k_t is a predetermined variable.

Let us notice that, using the definition of ω_{t+1} and substituting (19) into (11), we get $\eta_1(s_t) < 1/i_{t+1}$. Hence, at equilibrium, Assumption 2 implies:

$$1 < i_{t+1} < 1/\eta_1(s_t) \tag{25}$$

for $t = 0, 1, ..., +\infty$.

4 Steady state analysis

A steady state is a solution $(s, k) \in \mathbb{R}^2_{++}$ that satisfies:

$$x = \frac{1-a}{a}\frac{(1-\eta_1(s))\, r(k)}{\gamma(s) + \mu(1-\gamma(s)-\eta_1(s))\, r(k)/n} \tag{26}$$

with

$$x = \frac{r(k)}{\gamma(s)(w(k)/s - 1) - (1-\gamma(s))\, r(k)/n}$$

and[15]

$$r(k)(s - nk) = n(s - nk) \tag{27}$$

By direct inspection of equation (27), we deduce that two steady states may coexist, the one without bubble (bubbleless steady state), where $s = nk$, and the one with a bubble (bubbly steady state), where $s > nk$.

For the sake of brevity, we will omit the characterization of the former. Indeed, the novelty of the paper mainly rests on the role of monetary policy and credit market regulation[16] on the level of the bubble and the occurrence of persistent fluctuations of the bubble.

Using (26) and (27), a steady state with $s > nk$ is a solution $(s, k) \in \mathbb{R}^2_{++}$ satisfying:

$$r(k) = n \tag{28}$$

$$\frac{a}{1-a}\frac{ns/\gamma(s)}{w(k) - s/\gamma(s)} = \frac{n[1-\eta_1(s)]}{\gamma(s) + \mu[1-\gamma(s)-\eta_1(s)]} \tag{29}$$

Equation (28) determines the capital intensity of golden rule, which, in turn, determines the wage bill $w(k)$. Replacing $w(k)$ in (29) gives the non-monetary savings s as a function of the efficient capital intensity.

At the steady state, equation (15) writes $\mu = n\pi$ and gives, together with equation (17), the Fischer equation of a bubbly regime: $i = \pi n$. Therefore,

[15] Equation (27) is equivalent to $r(k)\, b = nb$.

[16] Recall that the credit share $\gamma(s)$ summarizes either lenders' habits based on the existence of asymmetric information about borrowers, or institutional and legal constraints to loans.

according to equation (25), Assumption 2 holds if and only if the monetary policy is bounded:

$$1 < \mu < 1/\eta_1(s) \tag{30}$$

In the following, after tackling the issue of the existence of a bubbly steady state, we will pursue the analysis by studying how money growth and credit share affect the stationary allocation. We end the section by evaluating the consequences of monetary policy on welfare.

4.1 Existence

The following assumption is sufficient to ensure the existence of a steady state with a positive bubble:

Assumption 4

$$\frac{a\alpha}{\gamma\left(nf'^{-1}(n)\right)(1-\alpha)-\alpha} < \frac{(1-a)\left[1-\eta_1\left(nf'^{-1}(n)\right)\right]}{\mu\left[1-\eta_1\left(nf'^{-1}(n)\right)\right]-(\mu-1)\gamma\left(nf'^{-1}(n)\right)}$$

where $\alpha \equiv \alpha\left(f'^{-1}(n)\right)$ is the capital share in total income at the golden rule.

The next proposition proves the existence and provides a result on uniqueness.

Proposition 1 *Let $\underline{s} \equiv nf'^{-1}(n)$ and $\overline{s}$ be defined by $w \equiv w\left(f'^{-1}(n)\right) = \overline{s}/\gamma(\overline{s})$. Under Assumptions 1-4, there exists a steady state characterized by the golden rule, $r(k) = n$, and a positive bubble, $s \in (\underline{s}, \overline{s})$. Moreover, when γ is constant, uniqueness of this steady state is ensured.*

Proof. See the Appendix.

By continuity, uniqueness of the steady state with bubble is still satisfied when the credit share $\gamma(s)$ is no longer constant but the elasticity of credit share $\eta_1(s)$ remains sufficiently weak for every $s \in (\underline{s}, \overline{s})$.

4.2 Comparative statics under isoelastic credit share

For simplicity, we focus on the case with a constant elasticity of credit share η_1, that is $\eta_\eta(s) = 0$. We start by studying the role of the credit constraint on non-monetary savings and, then, on the size of the bubble. The following assumption simplifies the comparative statics to a large extent.

Assumption 5

$$\mu < 1 + \frac{(1-\eta_1)^2}{\eta_1}\frac{1-a}{a}\frac{w}{s}$$

This assumption is not too restrictive. In particular, when we assume a sufficiently low η_1, to avoid large distortions in the credit market, Assumption 5 is easily satisfied.

Under an isoelastic credit share, the following proposition sheds a light on the structure of the credit market and the effects on the size of the bubble.

Proposition 2 *Under Assumptions 1-5 and a constant η_1, non-monetary savings s and the bubble b are both increasing in η_1.*

Proof. See the Appendix.

Under a positive (but not too large) rate of money growth $\mu - 1$ (see (30) and Assumption 5), the more sensitive the credit share to collaterals, the higher the non-monetary saving. Indeed, under a more elastic credit share, increasing s results in a larger amount on the right-hand side of the cash-in-advance constraint $c_2 \leq nm/[1 - \gamma(s)]$. Future consumption c_2 is allowed to rise. This, in turn, promotes and reinforces the initial rise of s. Since η_1 affects neither the capital-labor ratio, nor the inflation, the more sensitive credit share to collaterals also increases the size of the bubble.

To understand the role of the credit share on non-monetary savings, we further assume a constant credit share: $\eta_1 = 0$. Under a positive monetary growth ($\mu > 1$), γ has an unambiguously positive effect on the non-monetary savings s through the positive impact on their bubbly part b.[17] Indeed, households are required to hold less cash, enlarging the non-monetary part in total savings. Since capital intensity is fixed by the golden rule ($r(k) = n$), the non-monetary savings are forced to grow through a larger bubble.

Even if in presence of bubbles, because of the golden rule, the monetary policy fails to affect the capital-labor ratio, the non-monetary savings are actually modified by the money growth because of its effect on the bubble. Under the terms of Assumption 5, the following proposition explains the role of monetary policy on saving behavior.

Proposition 3 *Under Assumptions 1-5 and a constant η_1, non-monetary savings s are decreasing in μ if and only if $\eta_1 < 1 - \gamma$.*

Proof. See the Appendix.

A higher μ increases the inflation rate and the nominal interest rate, i.e. the opportunity cost of holding money. This reduces the demand of real balances. When the credit share is little sensitive to collaterals ($\eta_1 < 1 - \gamma$), the cash-in-advance constraint lowers the future consumption which, according to the

[17] When γ is constant, we can replace $\eta_1 = 0$ in (29) and differentiate with respect to γ and s. Under $\mu > 1$, we find a positive elasticity:

$$\frac{ds}{d\gamma}\frac{\gamma}{s} = \frac{1 + a(\mu - 1)}{1 + a(\mu - 1)(1 - \gamma)} > 0$$

At the steady state, savings are linear in the bubble: $s = \mu b + nf'^{-1}(n)$, and we obtain also a positive effect on the bubble: $db/d\gamma = (ds/d\gamma)/\mu > 0$.

budget constraint, requires less non-monetary saving ($\varepsilon_{s\mu} < 0$). Conversely, if credit market sensitivity to collaterals becomes large enough ($\eta_1 > 1 - \gamma$), individuals can reduce the burden of cash-in-advance by purchasing collaterals ($\varepsilon_{s\mu} > 0$), so offsetting the increase of nominal interest rate.[18]

It is also of interest to see how the (real) bubble b adjusts in response to a change of money growth. We can show that:

Corollary 1 *Under Assumptions 1-5 and a constant η_1, the (real) bubble b is decreasing in μ if $\eta_1 < 1 - \gamma$.*

Proof. See the Appendix.

When $\eta_1 < 1 - \gamma$, a higher rate of money growth reduces the size of the (real) bubble because it lowers non-monetary saving, but also because inflation rises. If $\eta_1 > 1 - \gamma$, a more expansive monetary policy can increase the size of the (real) bubble if the increase of non-monetary savings is sufficiently large. This occurs if, for instance, the credit share γ is sufficiently low.

4.3 Welfare

We conclude the steady state analysis by focusing on the role of monetary policy on consumers' welfare.

In the bubbly regime, as seen above, the capital intensity k of golden rule no longer depends on the monetary policy, whereas non-monetary savings s and, therefore, consumptions (when young and old) are affected by the choice of μ. More explicitly, c_1 and c_2 write:

$$c_1 = f(k) - nk - \frac{s}{\gamma(s)} \tag{31}$$

$$c_2 = n\frac{s}{\gamma(s)} \tag{32}$$

At the steady state, the individual welfare level is given by $W = U(c_1, c_2)$. Let:

$$\mu_1 \equiv \frac{\gamma}{\eta_1 - (1-\gamma)}$$

$$\mu_2 \equiv 1 + \frac{1-\eta_1}{1-\eta_1+\eta_\eta}\frac{(1-\eta_1)^2}{\eta_1}\frac{1-a}{a}\frac{w}{s}$$

After some computations, we obtain:[19]

$$\varepsilon_{W\mu} = \varepsilon_{Uc_2}\frac{\mu}{\gamma}\frac{1-\eta_1}{\eta_1}\frac{1-\eta_1}{1-\eta_1+\eta_\eta}\frac{\mu-1}{\mu-\mu_1}\frac{1-\gamma-\eta_1}{\mu-\mu_2} \tag{33}$$

[18] This interpretation is corroborated by the fact that the consumption ratio ($x \equiv c_2/c_1$) is increasing with respect to s (see the proof of Proposition 3). Therefore, the consumption ratio decreases (increases) with μ when $\eta_1 < 1 - \gamma$ ($\eta_1 > 1 - \gamma$).

[19] The welfare elasticity (33) is derived in the Appendix.

To characterize the welfare adjustment to the monetary policy, we further assume:[20]

Assumption 6 $\eta_\eta > \eta_1 - 1$.

In the next proposition, we highlight the welfare consequences of money growth ($\mu > 1$) depending on the credit market features.

Proposition 4 *Let Assumptions 1-4 and 6 be satisfied.*

1. *When* $\eta_1 < 1 - \gamma$, *the welfare* W *is decreasing for* $1 < \mu < \mu_2$ *and increasing for* $\mu > \mu_2$*;*
2. *When* $\eta_1 > 1 - \gamma$, *the welfare* W *is decreasing for* $1 < \mu < \min\{\mu_1, \mu_2\}$, *increasing for* $\min\{\mu_1, \mu_2\} < \mu < \max\{\mu_1, \mu_2\}$, *and decreasing again for* $\mu > \max\{\mu_1, \mu_2\}$.

In the limit case where $\mu = 1$, *the welfare* W *attains a local maximum.*

Proof. See the Appendix.

We know that a variation of μ induces a decrease or an increase of non-monetary savings s depending on the magnitude of η_1 relatively to $1 - \gamma$ (see Proposition 3). Moreover, by direct inspection of (31) and (32), we see that consumption demands c_1 and c_2 are, respectively, decreasing and increasing in s. Hence, when $\eta_1 < 1 - \gamma$ and μ is not too large, a higher rate of money growth, lowering non-monetary savings, results in a negative effect on welfare through the dominant contraction of second-period consumption. On the contrary, when $\eta_1 > 1 - \gamma$ and μ is not too large, welfare decreases with the money growth rate, because the rise of non-monetary savings comes from a lower first-period consumption with a dominant impact on welfare.

In any case, it is important to notice that, starting with a money growth rate which is not too large, decreasing μ is welfare improving.

Eventually, we observe that, in the limit case where μ tends to 1, credit market distortions no longer affect the consumer's choice. We recover on the one hand the Friedman rule ($i = n\pi = 1$) and, on the other hand, the intertemporal trade-off of a Diamond (1965) model without cash-in-advance corresponding to the golden rule, i.e. $U_1(c_1, c_2)/U_2(c_1, c_2) = r = n$ (see equation (26)).

5 Sunspot bubbles

Let us show the existence of sunspot bubbles, that is, multiple equilibria that converge to a steady state with a positive rational bubble. In order to address the issue, we will show that the steady state with a positive bubble can be locally indeterminate and, therefore, there is room for expectation-driven fluctuations of the bubble, without any shock on the fundamentals. Collaterals visibly

[20] The isoelastic case ($\eta_\eta = 0$) satisfies Assumption 6.

matter. Indeed, when the credit share is constant, the steady state is always determinate, while, when it depends on non-monetary savings, indeterminacy can arise under arbitrarily weak market distortions.

We start by linearizing the dynamic system (22)-(24) around the steady state with a positive bubble[21] and we obtain a preliminary lemma.

Lemma 2 *Let*

$$
\begin{aligned}
Z_1 &\equiv (1-\gamma-\eta_1)\left[\frac{1-a}{a}+\mu\frac{1-\gamma-\eta_1}{(1-\gamma)(1-\eta_1)}\right] && (34)\\
Z_2 &\equiv \gamma\left[\frac{\mu-1}{1-\eta_1}\left(1+\eta_1+\frac{\eta_1}{1-\eta_1}\eta_2\right)-\mu\frac{1-\gamma-\eta_1^2}{(1-\gamma)(1-\eta_1)}-\frac{1-a}{a}\right] && (35)\\
Z_3 &\equiv \frac{1-a}{a}\left(1+\eta_1\frac{1-y}{y}\right)+\mu\frac{1-\eta_1-\gamma}{1-\eta_1}\left(1+\frac{\eta_1}{1-\gamma}\frac{1-y}{y}\right) && (36)
\end{aligned}
$$

where the capital share in total non-monetary saving $y \equiv rk/(rk+ib) = nk/s \in (0,1]$ *and the credit market features* $\gamma \equiv \gamma(s)$, $\eta_1 \equiv \eta_1(s)$ *and* $\eta_2 \equiv \eta_2(s)$, *are all evaluated at the steady state.*

Under Assumptions 1-4, the characteristic polynomial, evaluated at a steady state with a positive bubble ($r(k) = n$, $y \in (0,1)$), *writes* $P(X) \equiv X^2 - TX + D = 0$, *where:*

$$
\begin{aligned}
D &= \frac{Z_1}{Z_2} - \frac{1-\alpha}{\sigma}\frac{Z_3}{Z_2} \equiv D(\sigma) && (37)\\
T &= 1 + D(\sigma) - \frac{1-\alpha}{\sigma}\frac{1-y}{y}\left(\frac{Z_1}{Z_2}-1\right) \equiv T(\sigma) && (38)
\end{aligned}
$$

Proof. See the Appendix.

Following Grandmont et al. (1998), we characterize the (local) stability properties of the steady state in the (T, D)-plane (see Figures 1 and 2). More explicitly, we evaluate the polynomial $P(X) \equiv X^2 - TX + D = 0$ at -1, 0 and 1. Along the line (AC), one eigenvalue is equal to 1, i.e. $P(1) = 1 - T + D = 0$. Along the line (AB), one eigenvalue is equal to -1, i.e. $P(-1) = 1 + T + D = 0$. On the segment $[BC]$, the two eigenvalues are complex and conjugate with unit modulus, i.e. $D = 1$ and $|T| < 2$. Therefore, inside the triangle ABC, the steady state is a sink, i.e. locally indeterminate ($D < 1$ and $|T| < 1 + D$). It is a saddle point if (T, D) lies on the right or left sides of both the lines (AB) and (AC) ($|1 + D| < |T|$). It is a source otherwise. Moreover, continuously changing a parameter of interest, we can follow how (T, D) moves in the (T, D)-plane. A (local) bifurcation arises when at least one eigenvalue crosses the unit circle, that is, when the pair (T, D) crosses one of the loci (AB), (AC) or $[BC]$. According to the changes of the bifurcation parameter, a pitchfork bifurcation (generically)

[21] The novelty of the paper concerns dynamics around the bubbly steady state. Thus, for the sake of conciseness, we omit the analysis of local dynamics in the neighborhood of the bubbleless steady state.

occurs when (T, D) goes through (AC),[22] a flip bifurcation (generically) arises when (T, D) crosses (AB), whereas a Hopf bifurcation (generically) emerges when (T, D) goes through the segment $[BC]$.

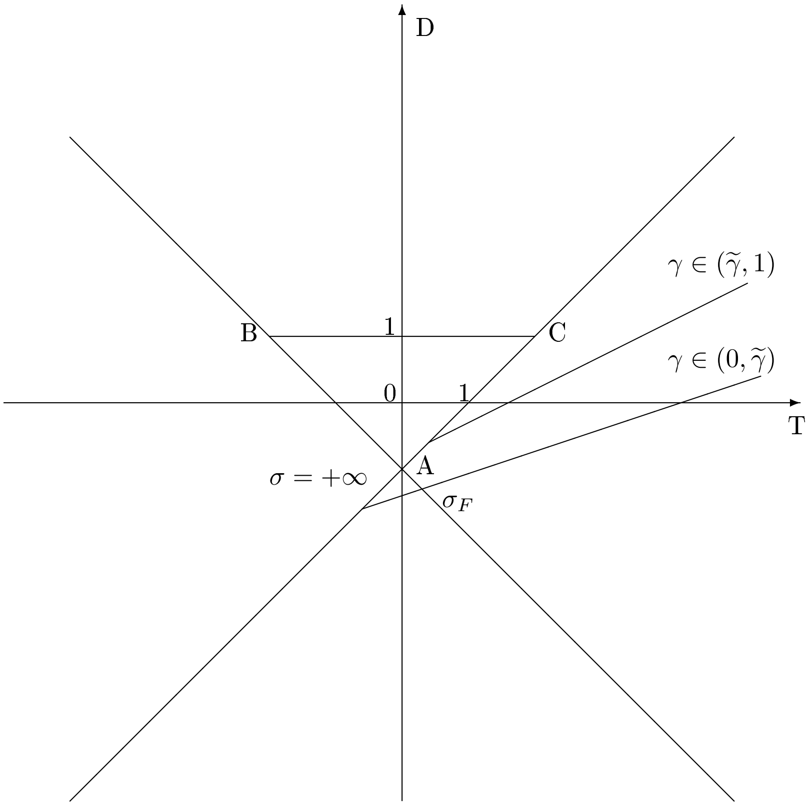

Figure 1: Local dynamics when γ is constant

A convenient parameter to discuss the stability of the steady state and the occurrence of bifurcations in the (T, D)-plane is the elasticity of capital-labor substitution $\sigma \in (0, +\infty)$. When this bifurcation parameter varies, the locus $\Sigma \equiv \{(T(\sigma), D(\sigma)) : \sigma \geq 0\}$ describes a half-line with a slope given by:

$$S = \frac{D'(\sigma)}{T'(\sigma)} = \frac{Z_3}{Z_3 + (Z_1 - Z_2)(1 - y)/y} \tag{39}$$

We notice also that the endpoint $(T(+\infty), D(+\infty))$ of the half-line Σ is located on the line (AC) and given by:

$$D(+\infty) = Z_1/Z_2 \text{ and } T(+\infty) = 1 + D(+\infty)$$

[22]Indeed, we have shown that there exists at least one steady state and the number of stationary solutions is generically odd (see Proposition 1).

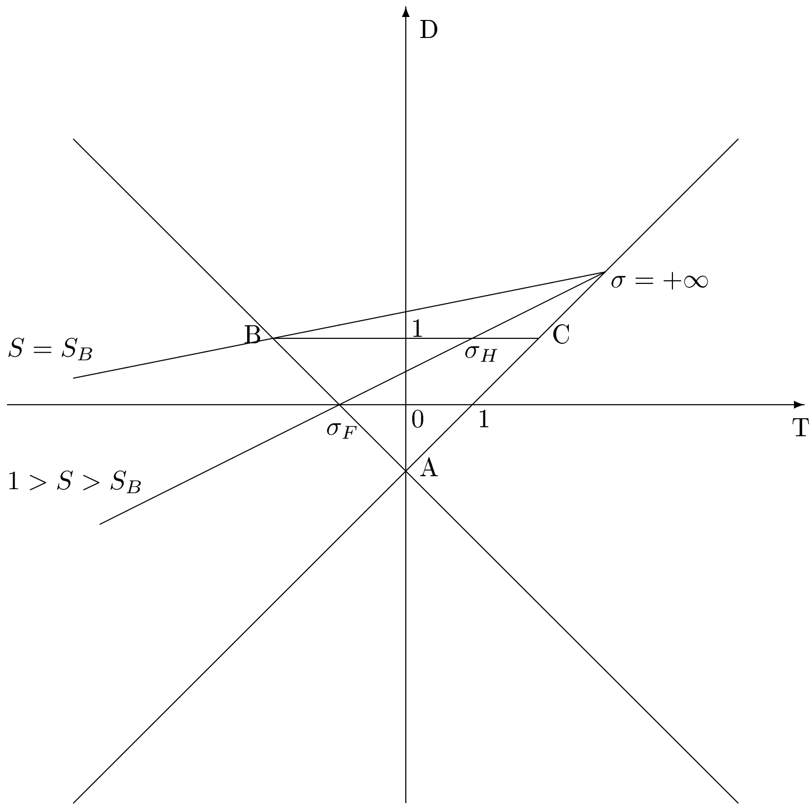

Figure 2: Indeterminate bubble

while, the starting point $(T(0^+), D(0^+))$ is such that $T(0^+) = \pm\infty$ and $D(0^+) = \pm\infty$, depending on the slope S.

In order to understand the role played by collaterals, we start by considering the case of a constant credit share: $\eta_1 = \eta_2 = 0$. Using equations (34)-(36), we get:

$$\frac{Z_1}{Z_2} = -\frac{1-\gamma}{\gamma}(a\mu + 1 - a) < 0 \text{ and } \frac{Z_3}{Z_2} = -\frac{1-\gamma}{\gamma}\left(a\mu + \frac{1-a}{1-\gamma}\right) < 0$$

Hence, the slope S belongs to $(0, 1)$ and $D(\sigma)$ is decreasing. This also means that $T(0^+) = +\infty$ and $D(0^+) = +\infty$. Moreover, since $D(+\infty) = Z_1/Z_2 < 0$, $(T(+\infty), D(+\infty))$ is on the line (AC) below the horizontal axis. Let:

$$\tilde{\gamma} \equiv \frac{1 + a(\mu - 1)}{2 + a(\mu - 1)} \in \left(\frac{1}{2}, 1\right) \tag{40}$$

We easily deduce that for $\gamma \in (\tilde{\gamma}, 1)$, $D(+\infty) > -1$, whereas for $\gamma \in (0, \tilde{\gamma})$, $D(+\infty) < -1$. Therefore, for $\gamma \in (\tilde{\gamma}, 1)$, the half-line Σ is below (AC) and above (AB). For $\gamma \in (0, \tilde{\gamma})$, Σ is still below (AC) but crosses (AB) at $\sigma = \sigma_F$, with:[23]

$$\sigma_F \equiv (1-\alpha)\left[\frac{a\mu(1-\gamma)+1-a}{(a\mu+1-a)(1-\gamma)-\gamma}+\frac{1}{2}\frac{1-y}{y}\frac{(a\mu+1-a)(1-\gamma)+\gamma}{(a\mu+1-a)(1-\gamma)-\gamma}\right] \tag{41}$$

Using these geometrical results, we deduce the following proposition:

Proposition 5 *Let $\tilde{\gamma}$ be defined by (40), σ_F by (41), γ be constant and $\eta_1 = \eta_2 = 0$. Under Assumptions 1-4, the following generically holds.*

(i) When $\gamma \in (\tilde{\gamma}, 1)$, the bubbly steady state is a saddle for all $\sigma > 0$.

(ii) When $\gamma \in (0, \tilde{\gamma})$, the bubbly steady state is a saddle for $0 < \sigma < \sigma_F$, undergoes a flip bifurcation at $\sigma = \sigma_F$ and becomes a source for $\sigma > \sigma_F$.

On the one side, when the credit share γ is constant, there is no room for local indeterminacy and expectation-driven fluctuations are ruled out. When γ is sufficiently large, the bubbly steady state is a saddle for all degrees of capital-labor substitution. This result is similar to Tirole (1985), we recover by taking the limit case as γ tends to 1. In contrast, when γ is weaker and the capital-labor substitution becomes large enough, the bubbly steady state looses the saddle-path stability through the occurrence of cycles of period two.[24]

On the other side, assuming a credit share sensitive to collaterals ($\eta_1 \neq 0$, $\eta_2 \neq 0$) can entail serious effects on the stability properties. More precisely, not only we will show that the steady state may be locally indeterminate and expectations-driven fluctuations of the (rational) bubble may occur, but also that such fluctuations appear under arbitrarily weak market distortions, that is, η_1 close to zero and γ close to one.

In order to get local indeterminacy, we require the half-line Σ to enter the triangle ABC (see Figure 2). More explicitly, $D(\sigma) > T(\sigma) - 1$ is a necessary condition to be inside ABC. Using (37) and (38), this inequality is equivalent to $Z_1/Z_2 > 1$, but this implies that $(T(+\infty), D(+\infty))$ lies on the line (AC) above the point C. Hence, Σ goes through ABC and local indeterminacy arises if the following two conditions are met:

1. $D(\sigma)$ is increasing;

2. $S_B < S < 1$, where $S_B \equiv (Z_1 - Z_2)/(Z_1 + 3Z_2) \in (0, 1)$ is the value of the slope S such that the half-line Σ goes through the point B.

[23] The critical value σ_F solves $D(\sigma_F) = -T(\sigma_F) - 1$.

[24] Conversely, in a cash-in-advance Ramsey model where $1-\gamma$ denotes the consumption share holding real balances, dynamics are three-dimensional and indeterminacy arises for sufficiently large γ (close to one) whatever the elasticity of intertemporal substitution, while one-dimensional saddle-path stability prevails for smaller credit shares (see Bosi and Magris (2003) for details). As we shall see, we obtain closely related results in our overlapping generations model when the credit share is no more constant.

Notice that $D'(\sigma) > 0$ is equivalent to $Z_3/Z_2 > 0$, which, together with $Z_1/Z_2 > 1$, ensures that $0 < S < 1$. In addition, $Z_3/Z_2 > 0$ and $Z_1/Z_2 > 1$ imply $T(0^+) = -\infty$ and $D(0^+) = -\infty$.

All these geometrical results are summarized in the following proposition:

Proposition 6 *Let*

$$\begin{aligned}
\sigma_F &\equiv (1-\alpha)\frac{2Z_3 + (Z_1 - Z_2)\frac{1-y}{y}}{2(Z_1+Z_2)} \\
\sigma_H &\equiv (1-\alpha)\frac{Z_3}{Z_1 - Z_2}
\end{aligned}$$

be the critical values of the capital-labor substitution such that $D(\sigma_F) = -T(\sigma_F) - 1$ *and* $D(\sigma_H) = 1$, *respectively.*

Under Assumptions 1-4, the steady state with a positive bubble is locally indeterminate if the conditions (i) $Z_1/Z_2 > 1$, *(ii)* $Z_3/Z_2 > 0$ *and (iii)* $S > (Z_1 - Z_2)/(Z_1 + 3Z_2)$ *are satisfied, where* Z_1, Z_2, Z_3 *are given by (34)-(36), and* S *by (39).*

In this case, local indeterminacy occurs for $\sigma \in (\sigma_F, \sigma_H)$. *Generically, the steady state undergoes a flip bifurcation at* $\sigma = \sigma_F$ *and a Hopf bifurcation at* $\sigma = \sigma_H$.

We remark that, since $0 < \sigma_F < \sigma_H < +\infty$, there is no room for a locally indeterminate bubble when the production factors are either too weak substitutes (σ sufficiently close to zero) or too large substitutes (σ high enough).

In order to provide an intuition of Proposition 6, we need to write conditions (i)-(iii) in terms of those structural parameters that capture the peculiarities of the model. Since we are interested in the effects of monetary policy and the credit market distortions, it is appropriate to focus on the money growth rate μ and the credit market features (γ, η_1, η_2).

Let us introduce the following critical values:

$$\begin{aligned}
\underline{\mu} &\equiv \frac{1-\gamma}{\eta_1}\frac{1+\eta_1+(1-\eta_1)\frac{1-a}{a}}{1+\eta_1-\gamma} \\
\overline{\mu} &\equiv 1 + \left[a\left(\frac{\gamma}{1-\eta_1}\frac{1+\eta_1}{1-\eta_1} - 1\right)\right]^{-1} \\
\theta_1 &\equiv -\frac{1-\eta_1^2}{\eta_1}\left[1 - \frac{1-\eta_1}{1+\eta_1}\left(1 + \frac{\frac{1}{a} + \frac{\mu\eta_1}{1-\eta_1}\frac{1-\gamma-\eta_1}{1-\gamma}}{\mu - 1}\right)\right] \\
\theta_2 &\equiv -\frac{1-\eta_1^2}{\eta_1}\left[1 - \frac{1-\eta_1}{1+\eta_1}\left(1 + \frac{\frac{1}{a} - \frac{M}{1-\eta_1}}{\mu-1}\right)\frac{1-\eta_1}{\gamma}\right]
\end{aligned}$$

where

$$M \equiv 2Z_3\frac{y}{1-y}\left(\sqrt{1+\frac{Z_1}{Z_3}\frac{1-y}{y}} - 1\right) \tag{42}$$

Let us put additional restrictions to find suitable conditions for local indeterminacy.

Assumption 7 *$\gamma < 1 - \eta_1$ and $\underline{\mu} < \mu < \overline{\mu}$.*

In order to show that there is a nonempty subset P_0 of the parameter space, satisfying Assumption 7, we observe that, when γ lies in a left neighborhood of $1 - \eta_1$, we have $0 < \underline{\mu} < \overline{\mu}$. Indeed, $\lim_{\gamma \to 1-\eta_1} \left(\overline{\mu} - \underline{\mu}\right) = 0$ and $\partial \left(\overline{\mu} - \underline{\mu}\right) / \partial\gamma|_{\gamma=1-\eta_1} = -(1-a)/(2a\eta_1) < 0$ imply that the interval $\left(\underline{\mu}, \overline{\mu}\right)$ becomes nonempty as soon as γ decreases from $1 - \eta_1$.

To prove that a nonempty subset $P_1 \subseteq P_0$ meets also Assumption 2, we require inequalities (30) to hold when $\mu \in \left(\underline{\mu}, \overline{\mu}\right)$. This happens for η_1 sufficiently close to zero because $1 < \underline{\mu}$, and $\overline{\mu} < 1/\eta_1$. Finally, there exists a nonempty subset $P_2 \subseteq P_1$ where the second-order conditions for utility maximization are verified: consider, for instance, arbitrarily weak credit market imperfections, that is, a sufficiently low elasticity η_1 and a sufficiently large γ (say, close to one).

Proposition 6 can be now revisited regarding the credit market features:

Proposition 7 *Under Assumption 7, the conditions (i)-(iii) of Proposition 6 are satisfied if*

$$\max\{\theta_1, \theta_2\} < \eta_2 \tag{43}$$

Proof. See the Appendix.

The proof of Proposition 7 shows also that $\max\{\theta_1, \theta_2\}$ is negative and, so, the admissible interval for the second-order elasticity of credit share η_2 admits negative values, provided that $\mu > \underline{\mu}$ is sufficiently close to $\overline{\mu}$.

This proposition shows that, when collaterals matter ($\eta_1 \neq 0$), endogenous cycles can occur not only through a flip bifurcation (cycle of period 2) but also through a Hopf bifurcation, which promotes the emergence of an invariant closed curve around the steady state.

Moreover, the steady state can be locally indeterminate: expectation-driven fluctuations of the bubble can arise around the (bubbly) steady state. Following Greenspan's words, agents' rational exuberance is interpreted as a volatility of rational expectations which drives persistent fluctuations of a rational bubble.

To the best of our knowledge, this result is new and rests on the existence of arbitrarily small market distortions, i.e. a sufficiently low elasticity of credit share (η_1 close to zero) together with large credit opportunities (γ close to one)).

Furthermore, local indeterminacy requires intermediate values of the elasticity of capital-labor substitution, neither too low nor too high (see Proposition 6). So, usual specifications of technology becomes compatible with the existence of multiple equilibria. Namely, a Cobb-Douglas technology is represented by a unit elasticity and local indeterminacy requires $\sigma_F < 1 < \sigma_H$, which is equivalent to:

$$\frac{Z_1 - Z_2}{1-\alpha} < Z_3 < \frac{Z_1 + Z_2}{1-\alpha} - \frac{1}{2}\frac{1-y}{y}\left(Z_1 - Z_2\right) \tag{44}$$

The right-hand (left-hand) inequality in (44) corresponds to $\sigma_F < 1$ $(1 < \sigma_H)$. The right-hand inequality is satisfied for an appropriate choice of η_2, while the left-hand inequality is satisfied for η_1 sufficiently close to $1 - \gamma$.

Finally, we notice that Proposition 7 has also some implications for the monetary policy. Indeed, indeterminacy requires $\underline{\mu} < \mu < \overline{\mu}$ (Assumption 7). Therefore, choosing a money growth factor μ higher than $\overline{\mu}$ or lower than $\underline{\mu}$ rules out expectation-driven fluctuations. We argue however, that choosing μ smaller than $\underline{\mu}$ and even sufficiently close to 1 is better. Indeed, in such a case, the monetary authority does not only stabilize fluctuations due to self-fulfilling expectations, but also improves consumers' welfare at the steady state (see Proposition 4). We notice also that this result is in contrast to Michel and Wigniolle (2005) where a sufficiently expansive monetary creation is recommended to avoid fluctuations of the economy switching between a regime with a bubble and a regime with a binding cash-in-advance constraint.

5.1 Economic intuition

First, we will give a story for bubbly cycles of period two based on the emergence of non-monotonic trajectories (Proposition 5). Then, we will provide an economic interpretation for the occurrence of local indeterminacy, that is the existence of sunspot bubbles or rational exuberance (Proposition 6).

We start with the case where the credit share γ is constant ($\eta_1 = 0$), but strictly smaller than one. Assuming a decrease of the capital stock k_t from its steady state value, the real wage w_t becomes smaller and the real interest rate r_t higher. When the elasticity of capital-labor substitution is not too weak, this induces a lower level of $r_t s_{t-1}$. Since, using equation (19), we have:

$$m_t = r_t s_{t-1} \frac{1}{n} \frac{1-\gamma}{\gamma} \tag{45}$$

real money balances m_t decreases. As a direct implication, we also get a decrease of $\pi_{t+1} m_{t+1}$ (see equation (15)).

Using now (22) and (23) with γ constant and $\eta_1 = 0$, we obtain:

$$s_t = (1-a) w_t - (a\mu + 1 - a) r_t s_{t-1} \frac{1}{n} \frac{1-\gamma}{\gamma} \tag{46}$$

Since both w_t and $r_t s_{t-1}$ decrease, two opposite effects affect savings s_t. In particular, we note that the second effect comes from the decrease of money holding and, obviously, disappears in the limit case where the credit share γ tends to one.

Assuming that the second effect dominates, savings s_t increases. Using (21), we deduce that $i_{t+1} = \pi_{t+1} r_{t+1}$ decreases, meaning that the opportunity cost of holding money is reduced. Therefore, money balances m_{t+1} increases, which implies a decrease of inflation π_{t+1} because, as seen above, $\pi_{t+1} m_{t+1}$ reduces. From equation (45), this increase of the real money stock implies a raise of $r_{t+1} s_t$. When capital and labor are not too weak substitutes, capital k_{t+1}

becomes higher. Since the bubble $\pi_{t+1}b_{t+1}$ has the same return, it increases also.

This explains that, following a decrease of capital from the steady state, future capital goes in the opposite direction, explaining oscillations. When γ is constant and not too close to one, we have seen that instability emerges (see Proposition 5). We argue that this comes from two main effects: the strong impact of $r_t s_{t-1}$ on s_t (see (46)) and the proportional relationship between $r_t s_{t-1}$ and m_t (see (45)).

Conversely, local indeterminacy requires a variable γ closer to one (see Proposition 7). If, on the one hand the effect of $r_t s_{t-1}$ on s_t is lower (see (46)), on the other hand the relationship between m_t and $r_t s_{t-1}$ is no longer proportional and becomes nonlinear:

$$m_t = \frac{1}{n}\frac{1-\gamma(s_{t-1})}{\gamma(s_{t-1})} r_t s_{t-1} \tag{47}$$

Note that the elasticity of $[1-\gamma(s)]/\gamma(s)$ with respect to s is equal to $-\eta_1/(1-\gamma)$, which belongs to $(-1,0)$ and is quite small in absolute value under Assumption 7. Therefore, when $r_t s_{t-1}$ decreases, and s_{t-1} as well, the effect on m_t is dampened. In other words, two crucial channels for the occurrence of non-monotonic dynamics are weaker when $\eta_1 > 0$, which provides the intuition for local stability or indeterminacy of the bubbly steady state when collateral matters. Finally, we notice that equation (24) rewrites:

$$\pi_t b_t r_t = n\pi_{t+1}b_{t+1} \tag{48}$$

The oscillations just described above can be sustained by optimistic expectations on the future value of the bubble $\pi_{t+1}b_{t+1}$, meaning that consumers born in $t-1$ will (slightly) increase their share of savings through the bubble $\pi_t b_t$, which implies an effective increase of the bubble in the next period $\pi_{t+1}b_{t+1}$, since r_t also raises.

6 Conclusion

Could market volatility, what Greenspan calls exuberance, be compatible with agents' rationality? In order to give a positive answer, we extend the Tirole (1985) model with rational bubbles, to account for credit market imperfections. We consider an overlapping generations model, where a share of the second-period consumption is paid by cash, while savings are also used to buy productive capital and a pure bubble. Collateral matters because a higher level of non-monetary savings reduces this share of consumption financed by money balances.

In this framework, we show that the bubbly steady state can be locally indeterminate because of the role of collateral and, therefore, there is room for expectation-driven fluctuations of the bubble. We further notice that the existence of such fluctuations requires arbitrarily small market distortions. We finally recommend the monetary policy to be not too expansive in order to achieve a twofold objective, that is, to immunize the economy against endogenous fluctuations and to improve the welfare level (evaluated at the steady state).

All these results concern equilibria where money is a dominated asset and the cash-in-advance constraint is always binding. In a simpler model where collaterals play no role, Michel and Wigniolle (2003, 2005) are able to prove that the economy can experience cycles by switching between two regimes where, respectively, the liquidity constraint is binding or fails to hold with equality. Analyzing such dynamics in our model is left for future research.

7 Appendix

Proof of Lemma 1

We maximize the Lagrangian function:

$$\begin{aligned}
&U\left(c_{1t}, c_{2t+1}\right) \\
&+\lambda_{1t}\left(\tau_t + w_t - n\pi_{t+1}m_{t+1} - s_t - c_{1t}\right) \\
&+\lambda_{2t+1}\left(nm_{t+1} + r_{t+1}s_t - c_{2t+1}\right) \\
&+\nu_{t+1}\left(nm_{t+1} - \left[1-\gamma\left(s_t\right)\right]c_{2t+1}\right)
\end{aligned} \tag{49}$$

with respect to $(m_{t+1}, s_t, c_{1t}, c_{2t+1}, \lambda_{1t}, \lambda_{2t+1}, \nu_{t+1})$. Since $\lambda_{1t} = U_1\left(c_{1t}, c_{2t+1}\right) > 0$, then (7) becomes binding. Because

$$\begin{aligned}
\lambda_{2t+1} &= \lambda_{1t}\frac{1-\pi_{t+1}\gamma'\left(s_t\right)c_{2t+1}}{r_{t+1}-\gamma'\left(s_t\right)c_{2t+1}} \\
\nu_{t+1} &= \lambda_{1t}\left(\pi_{t+1} - \frac{1-\pi_{t+1}\gamma'\left(s_t\right)c_{2t+1}}{r_{t+1}-\gamma'\left(s_t\right)c_{2t+1}}\right)
\end{aligned}$$

strict positivity of λ_{2t+1} and ν_{t+1} requires

$$\pi_{t+1} > \frac{1-\pi_{t+1}\gamma'\left(s_t\right)c_{2t+1}}{r_{t+1}-\gamma'\left(s_t\right)c_{2t+1}} > 0$$

or, equivalently,

$$i_{t+1} > \frac{r_{t+1}-i_{t+1}\gamma'\left(s_t\right)c_{2t+1}}{r_{t+1}-\gamma'\left(s_t\right)c_{2t+1}} > 0 \tag{50}$$

Inequality $r_{t+1} - i_{t+1}\gamma'\left(s_t\right)c_{2t+1} > 0$ is equivalent to (11). Moreover, $i_{t+1} > 1$ implies $r_{t+1} - \gamma'\left(s_t\right)c_{2t+1} > r_{t+1} - i_{t+1}\gamma'\left(s_t\right)c_{2t+1} > 0$, which ensures that both inequalities in (50) hold. ■

Sufficient conditions for utility maximization

We compute the Hessian matrix of the Lagrangian function (49) with respect to $(\lambda_{1t}, \lambda_{2t+1}, \nu_{t+1}, c_{1t}, c_{2t+1}, s_t, m_{t+1})$:[25]

$$H \equiv \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 & -n\pi \\ 0 & 0 & 0 & 0 & -1 & r & n \\ 0 & 0 & 0 & 0 & \gamma - 1 & c_2\gamma' & n \\ -1 & 0 & 0 & U_{11} & U_{12} & 0 & 0 \\ 0 & -1 & \gamma - 1 & U_{12} & U_{22} & \nu\gamma' & 0 \\ -1 & r & c_2\gamma' & 0 & \nu\gamma' & \nu c_2\gamma'' & 0 \\ -n\pi & n & n & 0 & 0 & 0 & 0 \end{bmatrix}$$

In order to get a regular (i.e. strict) local maximum, we need to check the negative definition of H over the set of points satisfying the constraints. Let m and n denote the numbers of constraints and variables, respectively. If the determinant of H has sign $(-1)^n$ and the last $n - m$ diagonal principal minors have alternating signs, then the optimum is a regular local maximum. In our case $n = 4$ and $m = 3$. Therefore, we simply require $\det H > 0$, that is,

$$\begin{aligned} \det H &= -n^2 \Big[(\gamma - \pi [c_2\gamma' - r(1-\gamma)])^2 U_{11} \\ &\quad + 2(c_2\gamma' - r)(\gamma - \pi [c_2\gamma' - r(1-\gamma)]) U_{12} \\ &\quad + (c_2\gamma' - r)^2 U_{22} \\ &\quad - \nu\gamma [2\gamma'(c_2\gamma' - r) - \gamma c_2\gamma'']\Big] > 0 \end{aligned} \tag{51}$$

Using (9) and (8), we find $c_{2t+1}/r_{t+1} = s_t/\gamma(s_t)$. Substituting in (51) in order to satisfy (locally) the second order conditions, we require:

$$\begin{aligned} \det H &= -(nr)^2 \left[\zeta_0 + \zeta_1^2 U_{11} + 2\zeta_1 (\eta_1 - 1) U_{12} + (\eta_1 - 1)^2 U_{22} \right] \\ &= -(nr)^2 \left[\zeta_0 + \begin{bmatrix} \zeta_1 & \eta_1 - 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{12} & U_{22} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \eta_1 - 1 \end{bmatrix} \right] > 0 \end{aligned} \tag{52}$$

where

$$\begin{aligned} \zeta_0 &= \zeta_0 \equiv \nu\eta_1 [\eta_2 + 2(1 - \eta_1)] \frac{\gamma}{r}\frac{\gamma}{s} \\ \zeta_1 &= \zeta_1 \equiv \pi (1 - \gamma - \eta_1) + \frac{\gamma}{r} \end{aligned}$$

Condition (52) ensures the concavity in the utility maximization program under three constraints. We observe that the negative definiteness of U entails

$$\begin{bmatrix} \zeta_1 & \eta_1 - 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{12} & U_{22} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \eta_1 - 1 \end{bmatrix} < 0 \tag{53}$$

A sufficient condition, jointly with (53), is $\zeta_0 < 0$ or, equivalently, $\eta_2 \leq 2(\eta_1 - 1)$, that is a sufficient degree of concavity of the credit share.[26] It is also

[25] For simplicity, the arguments of the functions and the time subscripts are omitted.

[26] In the isoelastic case, the concavity of credit share is weak: $\eta_2 = \eta_1 - 1$, and $\zeta_0 > 0$. In order to meet the second-order conditions for local maximization, we need a sufficiently concave utility function.

useful to notice that the second order condition is satisfied under a sufficiently small elasticity of credit share η_1, which implies ζ_0 close to zero.

In the Cobb-Douglas case, $\zeta_0 + \zeta_1^2 U_{11} + 2\zeta_1 (\eta_1 - 1) U_{12} + (\eta_1 - 1)^2 U_{22} < 0$ becomes:

$$\nu\eta_1 (\eta_2 + 2(1-\eta_1)) \frac{\gamma}{n}\frac{\gamma}{s} < a(1-a) c_1{}^a c_2^{1-a} \left[\frac{\gamma + \mu(1-\gamma-\eta_1)}{nc_1} + \frac{1-\eta_1}{c_2}\right]^2 \tag{54}$$

■

Proof of Proposition 1

The capital-labor ratio k is determined by the golden rule $r(k) = n$ (see (28)). Using Assumption 3, there exists a unique solution to this equation, $k = f'^{-1}(n)$. This also determines the real wage $w(k) = w\left(f'^{-1}(n)\right) = w$. Then, s is a solution of $g(s) = h(s)$, with:

$$g(s) \equiv \frac{a}{1-a} x(s), \text{ where } x(s) \equiv \frac{ns/\gamma(s)}{w - s/\gamma(s)} \tag{55}$$

$$h(s) \equiv \frac{n[1-\eta_1(s)]}{\gamma(s) + \mu[1-\gamma(s)-\eta_1(s)]} \tag{56}$$

Since the steady state is characterized by a positive bubble ($b > 0$), we have $s > \underline{s}$. Moreover, because $\eta_1(s) < 1$, $s/\gamma(s)$ is increasing in s, which implies that $x(s) > 0$ requires $s < \overline{s}$. Therefore, all the stationary solutions s belong to $(\underline{s}, \overline{s})$.[27]

To prove the existence of a stationary solution s, we use the continuity of $g(s)$ and $h(s)$, which is ensured by $\gamma \in C^2$ (see Assumption 1). Using (55) and (56), we determine the boundary values of $g(s)$ and $h(s)$:

$$\lim_{s\to\underline{s}} g(s) = \frac{a}{1-a}\frac{n^2 k}{w\gamma(nk) - nk} > 0 \qquad \lim_{s\to\overline{s}} g(s) = +\infty$$
$$\lim_{s\to\underline{s}} h(s) = \frac{n[1-\eta_1(nk)]}{\gamma(nk)+\mu[1-\gamma(nk)-\eta_1(nk)]} > 0 \quad \lim_{s\to\overline{s}} h(s) = \frac{n[1-\eta_1(\overline{s})]}{\gamma(\overline{s})+\mu[1-\gamma(\overline{s})-\eta_1(\overline{s})]}$$

where $k = f'^{-1}(n)$.

Assumption 4 ensures that $\lim_{s\to\underline{s}} g(s) < \lim_{s\to\underline{s}} h(s)$, while we have $\lim_{s\to\overline{s}} g(s) > \lim_{s\to\overline{s}} h(s)$. Therefore, there exists at least one value $s^* \in (\underline{s}, \overline{s})$ such that $g(s^*) = h(s^*)$.

To address the uniqueness versus the multiplicity of stationary solutions s, we compute the following elasticities:

$$\varepsilon_g(s) \equiv \frac{g'(s)s}{g(s)} = \frac{w[1-\eta_1(s)]}{w - s/\gamma(s)} > 0$$
$$\varepsilon_h(s) \equiv \frac{h'(s)s}{h(s)} = \frac{\eta_1(s)[\eta_\eta(s) + 1 - \eta_1(s)]}{1-\eta_1(s)} \frac{(\mu-1)\gamma(s)}{\gamma(s)+\mu[1-\gamma(s)-\eta_1(s)]}$$

[27] We notice that $(\underline{s}, \overline{s})$ is nonempty. Using (10) and (18), we obtain $w > s \geq nf'^{-1}(n) = \underline{s}$. Because $1-\eta_1(s)$, the elasticity of $s/\gamma(s)$, belongs to $(0,1)$, $s < w$ implies $s < \overline{s}$. We deduce that $\underline{s} < \overline{s}$.

A sufficient condition for uniqueness is $\varepsilon_h(s) < \varepsilon_g(s)$ for all $s \in (\underline{s}, \overline{s})$. We deduce that when $\gamma(s)$ is constant ($\eta_1(s) = 0$), uniqueness is ensured because $\varepsilon_h(s) = 0 < \varepsilon_g(s)$. ■

Proof of Proposition 2

Assuming η_1 constant and differentiating (29) with respect s and η_1, we obtain:

$$\varepsilon_{s\eta_1} \equiv \frac{ds}{d\eta_1}\frac{\eta_1}{s} = -\left(\frac{1-\eta_1}{\eta_1}\left[\eta_1 + (1-\eta_1)\frac{w}{s}\frac{1-\eta_1}{1-\mu}\frac{1-a}{a}\right]\right)^{-1}$$

Under (30) and Assumption 5, $\varepsilon_{s\eta_1} > 0$. According to (10), (21) and (28), we have $b = \left[s - nf'^{-1}(n)\right]/\mu$ and, hence, $db/d\eta_1 = (ds/d\eta_1)/\mu$. We easily conclude that b is also increasing in η_1. ■

Proof of Proposition 3

We differentiate (29) with respect to μ and s. Using $\eta_\eta = 0$, we obtain:

$$\varepsilon_{s\mu} \equiv \frac{ds}{d\mu}\frac{\mu}{s} = \frac{\mu}{\gamma}\frac{1-\gamma-\eta_1}{\eta_1(\mu-1) - (1-\eta_1)^2\frac{w}{s}\frac{1-a}{a}}$$

Since, under Assumption 5, the denominator of the right-hand side is strictly negative, the proposition immediately follows. ■

Proof of Corollary 1

Differentiating $b = \left[s - nf'^{-1}(n)\right]/\mu$, we get:

$$\varepsilon_{b\mu} \equiv \frac{db}{d\mu}\frac{\mu}{b} = \frac{s}{\mu b}\varepsilon_{s\mu} - 1$$

Under Assumption 5, we easily deduce that $\varepsilon_{b\mu} < 0$ if $\eta_1 < 1 - \gamma$. ■

Derivation of equation (33)

Consider the welfare function $W = U(c_1, c_2)$ and define the following elasticities:

$$(\varepsilon_{W\mu}, \varepsilon_{Uc_2}, \varepsilon_{c_2\mu}) \equiv \left(\frac{\partial W}{\partial \mu}\frac{\mu}{W}, \frac{\partial U}{\partial c_2}\frac{c_2}{U}, \frac{dc_2}{d\mu}\frac{\mu}{c_2}\right)$$

We immediately get:

$$\varepsilon_{W\mu} = \varepsilon_{Uc_2}\varepsilon_{c_2\mu}\left(1 + x\frac{a}{1-a}\frac{dc_1/d\mu}{dc_2/d\mu}\right) \tag{57}$$

Differentiating now (31) and (32), we obtain:

$$\frac{dc_1}{d\mu} = -(1-\eta_1)\frac{1}{\gamma}\frac{ds}{d\mu} \tag{58}$$

$$\frac{dc_2}{d\mu} = n(1-\eta_1)\frac{1}{\gamma}\frac{ds}{d\mu} \tag{59}$$

Substituting (58) and (59) in (57) and noticing that $\varepsilon_{c_2\mu} = (1-\eta_1)\,\varepsilon_{s\mu}$, we get:

$$\varepsilon_{W\mu} = \varepsilon_{Uc_2}\varepsilon_{s\mu}\,(1-\eta_1)\left(1-\frac{a}{1-a}\frac{x}{n}\right) \tag{60}$$

Equations (29) implicitly defines s as function of μ. Applying the Implicit Function Theorem, we find the following elasticity:

$$\varepsilon_{s\mu} = \frac{\mu}{\gamma}\frac{1-\gamma-\eta_1}{\eta_1\,(\mu-1)\,\frac{1-\eta_1+\eta_\eta}{1-\eta_1} - (1-\eta_1)^2\,\frac{w}{s}\frac{1-a}{a}} \tag{61}$$

Substituting (61) in (60), we have:

$$\varepsilon_{W\mu} = \varepsilon_{Uc_2}\frac{\mu}{\gamma}\left(1-\frac{x}{n}\frac{a}{1-a}\right)\frac{1-\gamma-\eta_1}{(\mu-1)\,\frac{\eta_1}{1-\eta_1}\frac{1-\eta_1+\eta_\eta}{1-\eta_1} - (1-\eta_1)\,\frac{w}{s}\frac{1-a}{a}}$$

Using the critical values μ_1 and μ_2, we deduce equation (33). ■

Proof of Proposition 4

Under Assumption 6, equation (33) implies that $\varepsilon_{W\mu}$ has the same sign of:

$$\frac{\mu-1}{\mu-\mu_1}\frac{1-\gamma-\eta_1}{\mu-\mu_2} \tag{62}$$

We note first that under Assumption 6, we have $\mu_2 > 1$. By direct inspection of (62), we deduce that:

1. When $\eta_1 < 1-\gamma$, we have $\mu_1 < 0$ and $1 < \mu_2$. Then, $\varepsilon_{W\mu} > 0$ for $0 < \mu < 1$; $\varepsilon_{W\mu} < 0$ for $1 < \mu < \mu_2$; $\varepsilon_{W\mu} > 0$ for $\mu > \mu_2$.

2. When $1-\gamma < \eta_1$, we have $1 < \mu_1$ and $1 < \mu_2$. Then, $\varepsilon_{W\mu} > 0$ for $0 < \mu < 1$; $\varepsilon_{W\mu} < 0$ for $1 < \mu < \min\{\mu_1,\mu_2\}$; $\varepsilon_{W\mu} > 0$ for $\min\{\mu_1,\mu_2\} < \mu < \max\{\mu_1,\mu_2\}$: $\varepsilon_{W\mu} < 0$ for $\mu > \max\{\mu_1,\mu_2\}$.

Therefore, $\mu = 1$ corresponds to a local maximum ($\varepsilon_{W\mu} = 0$). We deduce the proposition taking in account that $\mu > 1$. ■

Proof of Lemma 2

We linearize the system (22)-(24) around a steady state (with or without bubble) with respect to $(k_t, s_{t-1}, k_{t+1}, s_t)$. We obtain:

$$Z_2\frac{ds_t}{s} = \varepsilon_r\left(\gamma y\frac{1-a}{a} + \frac{1-\gamma}{1-\gamma-\eta_1}Z_1\right)\frac{dk_t}{k} + Z_1\frac{ds_{t-1}}{s} \tag{63}$$

$$y\frac{n}{r}\frac{dk_{t+1}}{k} - \frac{n}{r}\frac{ds_t}{s} = [y-(1-y)\,\varepsilon_r]\,\frac{dk_t}{k} - \frac{ds_{t-1}}{s} \tag{64}$$

where

$$\begin{aligned} Z_1 &\equiv (1-\gamma-\eta_1)\left[\frac{1-a}{a}+\mu\frac{1-\gamma-\eta_1}{(1-\gamma)(1-\eta_1)}\right] \\ Z_2 &\equiv \left(\mu-\frac{n}{x}\frac{1-a}{a}\right)\left(1+\frac{\eta_1\eta_\eta}{1-\eta_1}\right)-\mu\gamma\frac{1-\gamma-\eta_1^2}{(1-\gamma)(1-\eta_1)}-\gamma\frac{n}{r}\frac{1-a}{a} \end{aligned}$$

and r, ε_r and η_η the stationary values of $r(k_t)$, $\varepsilon_r(k_t)$ and $\eta_\eta(s_t)$, respectively.

The characteristic polynomial is given by $P(X) \equiv X^2 - TX + D = 0$, where T and D represent the trace and the determinant of the Jacobian matrix, respectively. After some computations, we get:

$$D = \frac{1}{Z_2}\frac{r}{n}\left(Z_1\left[1+\varepsilon_r\frac{y(1-\gamma)+(1-y)\eta_1}{y(1-\gamma-\eta_1)}\right]+\varepsilon_r\gamma\frac{1-a}{a}\right) \tag{65}$$

$$T = \frac{r}{n}+\frac{n}{r}D+\varepsilon_r\frac{1-y}{y}\left(\frac{Z_1}{Z_2}-\frac{r}{n}\right) \tag{66}$$

The expressions given in the lemma are obtained when $y < 1$, setting $r = n$ and using

$$x = \frac{1-a}{a}\frac{n(1-\eta_1)}{\gamma+\mu(1-\gamma-\eta_1)}$$

■

Proof of Proposition 7

We prove that, under Assumption 7, condition (43) is sufficient for local indeterminacy, implying conditions *(i)-(iii)* of Proposition 6.

Assuming $Z_2 > 0$,[28] conditions *(i)-(iii)* for local indeterminacy in Proposition 6 are equivalent to $Z_1 > Z_2$, $Z_3 > 0$ and

$$Z_2^2 - 2\left(Z_1+2Z_3\frac{y}{1-y}\right)Z_2+Z_1^2<0 \tag{67}$$

that is, to $Z_3 > 0$ and $0 < Z_1 - Z_2 < M$, where M is given by (42).

The inequality $\theta_1 < \eta_2$ is equivalent to $Z_2 > 0$, while the assumption $\gamma < 1-\eta_1$ implies $Z_3 > 0$. Since $\underline{\mu} > 1$, we have $\overline{\mu} > 1$, that is,

$$(1-\eta_1)\frac{1-\eta_1}{1+\eta_1}<\gamma \tag{68}$$

According to $1 < \underline{\mu} < \mu$ and (68), $\mu < \overline{\mu}$ implies $0 < Z_1 - Z_2$, while $\theta_2 < \eta_2$ is equivalent to $Z_1 - Z_2 < M$. Moreover, we notice that $\underline{\mu} < \mu$ is equivalent to $\theta_1 < 0$ and

$$M > (\overline{\mu}-\mu)\frac{1+\eta_1}{1-\eta_1}\left[\gamma-(1-\eta_1)\frac{1-\eta_1}{1+\eta_1}\right] \tag{69}$$

which is satisfied for μ sufficiently close to $\overline{\mu}$, entails $\theta_2 < 0$. ■

[28] Conditions *(i)-(iii)* of Proposition 6 are no longer met when $Z_2 < 0$.

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